Complex expressions

1. Remember the polar form $z = re^{i\theta}$.
2. Let $\Sigma := \{(u, v, w) \mid u^2 + v^2 + (w - \frac{1}{2})^2 = \frac{1}{4}\}$. Stereographic projection $\varphi : \Sigma \setminus N \to \mathbb{C}$, where $N = (0, 0, 1)$, is given by
$$\varphi(u, v, w) = \left(\frac{u}{1-w}, \frac{v}{1-w}\right); \quad \varphi^{-1}(x, y) = \left(\frac{x}{x^2+y^2+1}, \frac{y}{x^2+y^2+1}, \frac{x^2+y^2}{x^2+y^2+1}\right).$$
3. $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ and $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ for $z \in \mathbb{C}$.
4. $\log z = \log r + i\theta + i(2\pi n)$.

Holomorphic functions

1. To show that a function f is holomorphic, use the Cauchy-Riemann equations and show that the partials are continuous. You can also any rule from real analysis (the chain rule, the product rule, the quotient rule, etc.), or show that f has a power series development.
2. Let $f(x, y) = u(x, y) + iv(x, y) := (u(x, y), v(x, y))$. Recall that the Cauchy-Riemann equations are
$$f_y(z) = if_x(z) \iff u_x = v_y, \quad \text{and} \quad u_y = -v_x.$$
3. f is holomorphic on an open set $\iff$ f complex analytic $\iff$ f smooth; this means you can substitute any holomorphic function with its power series, and can differentiate it endlessly.
4. The **closed curve theorem** tells us that the integral of a function that is holomorphic in an open disk (or an open polygonally simply connected region) D over a closed contour $C \subset D$ is 0.
5. The **antiderivative theorem**, if $f(z)$ is holomorphic on an open disk (or an open polygonally simply connected region) D, then it has an antiderivative $F(z)$ such that $F'(z) = f(z)$.
6. **Cauchys integral formula** allows us to determine the value of a holomorphic function f at a point. In particular,
$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w}\, dz,$$
where f is holomorphic in an open disk (or an open polygonally simply connected region) D and w is any point in the region bounded by the positively oriented $C \subset D$.
7. The **uniqueness theorem** tells us that holomorphic functions are determined uniquely up to their values in a certain region.
8. The **mean value theorem** allows us to evaluate a holomorphic function at point. If $f(z)$ is holomorphic on $D = D(p, R)$, then for any $0 < r < R$, we have
$$f(p) = \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{i\theta})\, d\theta.$$
9. The **maximum modulus principle** says that if f is holomorphic in any region, then its maximum modulus is achieved at the boundary.

10. The **minimum modulus principle** says that if f is a non-constant analytic function in a region D, then no point $z \in D$ can be a relative minimum of f unless $f(z) = 0$. For example if $f(z) = z$ for $z \in D = D(0; r)$, then $\min_D |f| = |f(0)| = 0$, the mimimum modulus is 0 and it is achieved at an interior point.

Entire functions

1. Entire functions are functions that are holomorphic in all of $\mathbb{C}$.
2. **Liouvilles theorem** tells us that any bounded entire function is constant. A generalization is that if $|f(z)| \leq A + B|z|^N$ and $f(z)$ is entire, then $f(z)$ is a polynomial of degree $\leq N$.
3. likewise, , if $f(z)$ is entire and $\lim_{z \to \infty} f(z) = \infty$, then $f(z)$ is a polynomial.
4. The **fundamental theorem of algebra** says that a nonconstant polynomial always has a root.

Others

1. Know the denitions of radius of convergence, $\overline{\lim}_{k \to \infty} |c_k|^{1/k} = L$, and convergence tests/arguments for power series. These may be things that range from proving that a power series of a holomorphic function is uniformly convergent to finding the radii of convergence of some functions.
2. In evaluating a nontrivial integral, you need to know how to the closed curve theorem to give a contour integral as zero, then decompose the contour and bound the individual parts to find a specic integral.
3. A helpful tool is that you can bound the absolute value of any integral over C_R, circle of radius R centered at 0, by $2\pi R$ times the integrand; this is known as the ML theorem or the estimation lemma.